

An announcement of a learning seminar on “The distribution of ℓ^∞ -Selmer groups in degree ℓ twist families”

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I will be hosting a learning seminar/discussion group for my work on the distribution of Selmer groups in twist families. This seminar will be based on my recent two part paper:

1. “The distribution of ℓ^∞ -Selmer groups in degree ℓ twist families,” [arXiv link](#)
2. “The distribution of fixed point Selmer groups in twist families”, [arXiv link](#).

The learning seminar will begin in early September and proceed for about two months. It will be held virtually. If you are interested in participating, please email me at asmith13@stanford.edu as soon as you can.

What do these papers do?

Among other consequences, these papers show that:

- Conditional on the Birch and Swinnerton–Dyer conjecture, almost every elliptic curve over \mathbb{Q} satisfies Goldfeld’s conjecture;
- A reasonable generalization of the Cohen–Lenstra–Gerth heuristics accurately describes the ℓ^∞ -part of the class group of cyclic degree ℓ extensions of a given number field that does not contain $\mu_{2\ell}$;
- Given an integer polynomial f of odd degree $n \geq 5$ whose Galois group is S_n , the equation $dy^2 = f(x)$ has at most 2 rational solutions for 100% of integers d ; and
- For any abelian variety A/\mathbb{Q} and any $C > 0$,

$$\limsup_{H \rightarrow \infty} \frac{1}{H} \sum_{0 < d < H} \exp\left(C \cdot \text{rank}\left(A/\mathbb{Q}(\sqrt{d})\right)\right) \text{ is finite.}$$

Goals of the seminar

The primary goal of this group is to assist the mathematicians who would like to understand the methods of these two papers. We will go through the first paper thoroughly and portions of the second paper less thoroughly.

Participants should expect to be active in the group and to do some reading and thinking outside the official meeting times. The first paper builds up on itself in a way that would make passive participation less rewarding.

Towards future research

The longer term goal of this seminar is to enable number theorists to prove new results by refining and extending the methods in my two papers. The recent work on Steinhilber's conjecture by Koymans and Pagano is not a fluke; there are many open arithmetic statistical questions where my methods look promising but there is some obstruction to handle. These obstructions come in many different sizes:

- (*Smaller obstructions*) Find the distribution of 2-Selmer ranks in the quadratic twist family of $y^2 = x(x - 1)(x - 4)$.
- (*Medium obstructions*) Find the distribution of 2^∞ -class groups in the family of quadratic extensions of $\mathbb{Q}(\sqrt{-1})$.
- (*Enormous obstructions*) Find the distribution of 3^∞ -Selmer ranks of the curves in the cubic twist family of $y^2 = x^3 - 432$.

At this point, more than six years into this project, I would rather have a supporting role in the push to solve these problems than work on them alone.

Format of the seminar

The logistics of the seminar are still subject to change, but here is the current version:

The seminar will meet once a week for about ninety minutes. It will take place on Zoom and Overleaf. I will not plan seminar talks; if material is presented, either by me or by participants, it will be presented in short bursts. Otherwise, the seminar will consist of discussion and questions. Each week, I will give out exercises that will be discussed at the next week's seminar.

I would recommend that the participants form study groups that meet outside the scheduled seminar. I will also have office hours once or twice a week.

I will make a version of the papers available on Overleaf. These will be open to editing and annotation so that people can ask questions, provide feedback, and clarify points in the text as they see fit. We will use these Overleaf documents together with something like Notability as the blackboard for the seminar discussion. I will take feedback on the Overleaf seriously; it will impact the published form of these articles.

The seminar will not be open to the public, and the discussion will not be recorded. You must email me if you are interested in participating.

Other questions?

If you have questions, please email me at asmith13@stanford.edu.